PHYSICAL REVIEW E **83**, 061114 (2011)

Entropic equation of state and scaling functions near the critical point in uncorrelated scale-free networks

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(Received 19 January 2011; published 13 June 2011)

We analyze the entropic equation of state for a many-particle interacting system in a scale-free network. The analysis is performed in terms of scaling functions, which are of fundamental interest in the theory of critical phenomena and have previously been theoretically and experimentally explored in the context of various magnetic, fluid, and superconducting systems in two and three dimensions. Here, we obtain general scaling functions for the entropy, the constant-field heat capacity, and the isothermal magnetocaloric coefficient near the critical point in uncorrelated scale-free networks, where the node-degree distribution exponent λ appears to be a global variable and plays a crucial role, similar to the dimensionality d for systems on lattices. This extends the principle of universality to systems on scale-free networks and allows quantification of the impact of fluctuations in the network structure on critical behavior.

DOI: 10.1103/PhysRevE.83.061114 PACS number(s): 64.60.aq, 64.60.F−, 75.10.−b

I. INTRODUCTION

Phase transitions and critical behavior in complex networks currently attract much attention [1] because of their unusual features and broad array of applications, ranging from socio- [2] to nanophysics [3]. It is by now well established that the critical behavior of a many-particle interacting system located on the nodes of a general network may crucially differ from that of a system located on the sites of a d-dimensional regular lattice. Of particular interest are the so-called scale-free networks, for which the probability to find a node of degree k (i.e., with k nearest neighbors) vanishes for large k as a power law,

$$P(k) \sim k^{-\lambda}. \tag{1}$$

The questions we address in this paper concern two fundamental principles of critical phenomena: universality and scaling [4]. Both of these questions have to be reconsidered when a system resides on a network. Usually, the universality of critical phenomena is understood as stating that the thermodynamic properties near the critical point T_c are governed by a small number of global features, such as dimensionality, symmetry, and the type of interaction. In turn, the scaling hypothesis states that the singular part of a thermodynamic potential near T_c has the form of a generalized homogeneous function. To be specific, for the Helmholtz potential of a magnetic system, the latter can be written as [5]

$$F(\tau,M) = \tau^{2-\alpha} f_{\pm}(M/\tau^{\beta}), \tag{2}$$

where M is the magnetization, $\tau = |T - T_c|/T_c$, α,β are the universal exponents, and the sign $\pm$ corresponds to $T > T_c$ or $T < T_c$, respectively. The essence of relation (2) is that the two-variable function $F(\tau,M)$, when appropriately rescaled, is expressed in terms of a single scaling variable leading to the *scaling function* $f_{\pm}(x)$. The expression (2) gives an example of the scaling function for the Helmholtz potential $F(\tau,M)$. Together with other scaling functions—for the equation of state and for thermodynamic functions—this appears to give a suitable and comprehensive description of critical phenomena [6,7]. These scaling functions are also universal in the sense explained above.

For systems on scale-free networks, the principle of universality is extended: there, the node-degree distribution exponent λ in Eq. (1) appears to be a global variable and plays a crucial role, similar to the dimension d for systems on lattices (see, e.g., [8–13]). All systems that belong to a given universality class are governed by the same values of critical exponents and critical amplitude ratios and share the same universal form of scaling functions. Recently, the scaling function formalism has been applied to describe the critical behavior of magnetic systems with the structure of uncorrelated scale-free networks [13], that is, networks that are maximally random under the constraint of a power-law degree distribution (1) (see [9]). There, scaling functions for the magnetic equation of state and isothermal susceptibility were derived. In this paper, we are interested in the entropic form of the equation of state. In particular, this opens the possibility to derive scaling functions for the heat capacity. These are of wide and fundamental interest in the theory of critical phenomena and have been the subject of thorough theoretical and experimental studies for various magnetic, fluid, and superconducting systems [14–21]. A particular point of interest concerns the existence or nonexistence of the moments of the distribution (1) and their influence on the universal behavior.

The setup of the paper is as follows: in Sec. II, we define the notations and derive expressions for the entropic equation of state, heat capacity, and magnetocaloric coefficient scaling functions for systems on scale-free networks. These

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expressions are further analyzed and compared with corresponding functions for bulk systems in Sec. III. The paper concludes with a summary and an outlook in Sec. IV.

II. FREE ENERGY AND SCALING FUNCTIONS

The critical behavior of a many-particle system interacting on a scale-free network depends crucially on the node degree distribution $P(k)$ via the decay exponent λ in Eq. (1) [1]. In particular, for an infinite network, the value of λ determines the order of the first diverging moment, this order being the lowest integer $j \geqslant \lambda - 1$. This is reflected by the phase-transition scenario. For low values of $\lambda \leqslant 3$, the system remains ordered for any finite temperature, whereas for $\lambda > 3$, a finite-temperature, order-disorder phase transition occurs. Moreover, critical exponents that govern a second-order phase transition in a scale-free network attain their usual mean-field values for high $\lambda > 5$ and demonstrate nontrivial λ dependence in the region $3 < \lambda < 5$. Logarithmic corrections to scaling laws appear at $\lambda = 5$: this resembles phenomena that occur at marginal space or order-parameter dimensions in bulk systems [22].

A starting point for our analysis will be the expression for the free energy of a system with scalar order parameter on a scale-free network. To be specific, from now on we will consider ferromagnetic ordering and the spontaneous magnetization M as the order parameter with the conjugate magnetic field H. In this case, the corresponding microscopic degrees of freedom are the Ising spins. However, generalization to models of more complicated symmetry is straightforward [11,12]. Due to the fact that the networks under discussion are assumed to have a local treelike structure, the mean-field approximation is asymptotically exact in the sense that thermal fluctuations can be neglected. This leads to a form of the free energy also found using other techniques. The lowest-order contributions to the singular part of the Helmholtz free energy in the vicinity of T_c are [8–10]

$$F(M,T) = \frac{a}{2}(T - T_c)M^2 + \frac{b}{4}M^4, \quad \lambda > 5, \tag{3}$$

$$F(M,T) = \frac{a}{2}(T - T_c)M^2 + \frac{b}{4}M^{\lambda-1}, \quad 3 < \lambda < 5. \tag{4}$$

The parameters $a,b > 0$ and the critical temperature T_c are λ-dependent. This dependence can be made explicit using microscopic approaches [8,9,11] or may be postulated in a Landau-like approach [10,12]. For the subsequent analysis, we will absorb the parameters into the dimensions of the corresponding observables, passing to dimensionless quantities,

$$f(m,\tau) = \pm\frac{\tau}{2}m^2 + \frac{1}{4}m^4, \quad \lambda > 5, \tag{5}$$

$$f(m,\tau) = \pm\frac{\tau}{2}m^2 + \frac{1}{4}m^{\lambda-1}, \quad 3 < \lambda < 5, \tag{6}$$

with obvious relations between dimension-dependent and dimensionless variables,

$$m = M/M_0, \quad \tau = |T - T_c|/T_c, \quad f = F/F_0, \tag{7}$$

where $M_0^2 = aT_c/b$ for $\lambda > 5$, $M_0^{\lambda-3} = aT_c/b$ for $3 < \lambda < 5$ and $F_0 = aT_cM_0^2$. Since τ measures the absolute distance to the critical point, the free energy has two branches, corresponding to signs "+" and "−" in Eqs. (5) and (6) for $T > T_c$ and $T < T_c$, respectively. It is easy to verify that a system with the free energy (5) and (6) possesses a second-order phase transition at $\tau = 0$. Here, we employ the standard notation for critical exponents governing the temperature and field dependencies of the thermodynamic functions. For $h = 0$ and $T \to T_c^{\pm}$, these are

$$c_h \simeq A^{\pm}\tau^{-\alpha}, \quad \chi_T \simeq \Gamma^{\pm}\tau^{-\gamma}, \quad m_T \simeq B_T^{\pm}\tau^{-\omega} \tag{8}$$

while for $T \to T_c^-$, one also has

$$m \simeq B\tau^{\beta}. \tag{9}$$

On the other hand, for $\tau = 0$, the standard definitions are

$$c_h \sim A_c h^{-\alpha_c}, \quad h \simeq D_c m|m|^{\delta-1}, \tag{10}$$

$$\chi_T \simeq \Gamma_c h^{-\gamma_c}, \quad m_T \simeq B_T^c h^{-\omega_c}. \tag{11}$$

(See Sec. II C for the definition of the magnetocaloric coefficient m_T.) The values of these critical exponents are summarized in Table I [8–11,13]. It is worth noting here that one way to derive the listed exponents is to consider the naive dimensions of different terms in the Landau free energy, similar to the standard field theoretical procedure (see, e.g., [23]). With the values of critical exponents at hand, one can rewrite the singular part of the Helmholtz potential in the form of a generalized homogeneous function (2) [4]:

$$f(m,\tau) = \tau^2 f_{\pm}(x), \quad x = m/\tau^{\frac{1}{2}}, \quad \lambda > 5, \tag{12}$$

$$f(m,\tau) = \tau^{\frac{\lambda-1}{\lambda-3}} f_{\pm}(x), \quad x = m/\tau^{\frac{1}{\lambda-3}}, \quad 3 < \lambda < 5, \tag{13}$$

where the free-energy scaling functions are given by [13]

$$f_{\pm}(x) = \pm\frac{1}{2}x^2 + \frac{1}{4}x^4, \quad \lambda > 5, \tag{14}$$

$$f_{\pm}(x) = \pm\frac{1}{2}x^2 + \frac{1}{4}x^{\lambda-1}, \quad 3 < \lambda < 5. \tag{15}$$

Assuming that the Helmholtz potential is a complete differential,

$$dF = -SdT + HdM, \tag{16}$$

one can further proceed with an analysis based on the magnetic form of the equation of state,

$$H = \frac{\partial F}{\partial M}\Big|_T, \tag{17}$$

or the entropic form of the equation of state [25],

$$S = -\frac{\partial F}{\partial T}\Big|_M. \tag{18}$$

As we have noted in the Introduction, the scaling functions for the magnetic equation of state (both in the Widom-Griffith [6] and Stanley-Hankey [24] forms) and isothermal susceptibility have recently been reported elsewhere [13]. Here, we will proceed by analyzing the entropic equation of state (18) and heat-capacity scaling functions.

In terms of dimensionless variables, Eqs. (17) and (18) take on the form

$$h(m,\tau) = \frac{\partial f(m,\tau)}{\partial m}\Big|_\tau, \quad s(m,\tau) = \mp\frac{\partial f(m,\tau)}{\partial \tau}\Big|_m \tag{19}$$

with field h and entropy s measured in units of F_0/M_0 and F_0/T_c, respectively. As before, and throughout, the index $\pm$ refers to temperatures above and below the critical point T_c.

Since the free energy (3) and (4) is explicitly a linear function of τ, one obtains the usual mean-field result for the heat capacity at constant magnetization:

$$C_M = T\frac{\partial S}{\partial T}\Big|_M = 0. \qquad (20)$$

To find the dimensionless constant-magnetic-field heat capacity [25],

$$c_h = \pm\frac{T}{T_c}\frac{\partial s(\tau,m)}{\partial \tau}\Big|_h = (\tau \pm 1)\frac{\partial s(\tau,m)}{\partial \tau}\Big|_h, \qquad (21)$$

one can consider the entropy as a function of magnetic field and temperature $s(\tau,m(\tau,h))$, which leads to

$$c_h = (\tau \pm 1)\Big[\frac{\partial s}{\partial \tau}\Big|_m + \frac{\partial s}{\partial m}\Big|_\tau \frac{\partial m(\tau,h)}{\partial \tau}\Big|_h\Big]. \qquad (22)$$

Noting from (3) and (4) that $\partial s/\partial m|_\tau = -m$ and $\partial s/\partial \tau|_m = 0$, one finally arrives at the expression for the heat capacity,

$$c_h = (1 \pm \tau)C_h(\tau,m), \qquad (23)$$

with the function C_h given by

$$C_h(\tau,m) = \mp m\frac{\partial m(\tau,h)}{\partial \tau}\Big|_h. \qquad (24)$$

Let us now consider separately the cases of fast ($\lambda > 5$) and slower ($3 < \lambda < 5$) decay of the node degree distribution (1).

A. $\lambda > 5$

The free energy (3) leads to the expression for the entropy,

$$s(\tau,m) = -\frac{m^2}{2}, \qquad (25)$$

which can be easily recast in a scaling form,

$$s(\tau,m) = \tau\mathcal{S}(x), \qquad (26)$$

where the scaling variable $x = m/\tau^\beta = m/\tau^{1/2}$ and the entropy scaling function $\mathcal{S}(x)$ is

$$\mathcal{S}(x) = -\frac{x^2}{2}. \qquad (27)$$

To obtain the heat capacity (22), we first write the magnetic equation of state (19),

$$h = \pm\tau m + m^3, \qquad (28)$$

and differentiate it with respect to τ to obtain

$$\frac{\partial m}{\partial \tau}\Big|_h = \frac{\mp m}{\pm\tau + 3m^2}. \qquad (29)$$

Substituting this into (24) leads to the representation of C_h in the form of a generalized homogeneous function,

$$C_h(\tau,m) = \mathcal{C}_\pm(x), \qquad (30)$$

with the scaling variable x defined above and the heat-capacity scaling function

$$\mathcal{C}_\pm(x) = \frac{x^2}{3x^2 \pm 1}. \qquad (31)$$

Note that in (30), the heat-capacity exponent vanishes, $\alpha = 0$.

B. $3 < \lambda < 5$

A particular feature of the entropy of a system on a scale-free network is that its dependence on magnetization both for $3 < \lambda < 5$ and for $\lambda > 5$ is given by Eq. (25). In terms of the scaling function for $3 < \lambda < 5$, it reads

$$s(\tau,x) = \tau^{2/(\lambda-3)}\mathcal{S}(x), \qquad (32)$$

where the entropy scaling function does not change and is given by Eq. (27). The power of τ is equal to $1-\alpha$ and the scaling variable is now

$$x \equiv m/\tau^\beta = m/\tau^{1/(\lambda-3)}. \qquad (33)$$

However, the magnetic equation of state (19) for the Helmholtz function (6) becomes λ-dependent:

$$h = \pm\tau m + \frac{\lambda-1}{4}m^{\lambda-2}. \qquad (34)$$

As in the previous subsection, we obtain from this the derivative $\partial m/\partial \tau|_h$, and by substitution into Eq. (24) we arrive at the representation of C_h in the form of the generalized homogeneous function,

$$C_h(\tau,m) = \tau^{\frac{5-\lambda}{\lambda-3}}\mathcal{C}_\pm(x), \qquad (35)$$

where the scaling variable x is given by (33) and the heat-capacity scaling function attains a nontrivial λ dependence,

$$\mathcal{C}_\pm(x) = \frac{x^2}{\frac{(\lambda-1)(\lambda-2)}{4}x^{\lambda-3} \pm 1}. \qquad (36)$$

Note that on the basis of the scaling functions $\mathcal{S}(x)$ in Eq. (27) and $\mathcal{C}_\pm(x)$ in Eqs. (31) and (36), one easily obtains the corresponding scaling functions with respect to the rescaled magnetic field,

$$y \equiv h/\tau^{\beta\delta}. \qquad (37)$$

The connection between the variables x and y results from the magnetic equations of state (28) and (34), and is given by

$$y = \pm x + x^3, \qquad \lambda > 5, \qquad (38)$$

$$y = \pm x + \frac{\lambda-1}{4}x^{\lambda-2}, \quad 3 < \lambda < 5. \qquad (39)$$

Solving the above equations with respect to x and substituting the result $x(y)$ into the functions $\mathcal{S}(x)$ and $\mathcal{C}_\pm(x)$ leads to the scaling functions $\mathcal{S}(y)$ and $\mathcal{C}_\pm(y)$. The behavior of the above scaling functions will be analyzed in the next section. These functions together with the scaling functions for the magnetic equation of state $h = \tau^{\beta\delta}H_\pm(m/\tau^\beta)$ and isothermal susceptibility $\chi_T = \tau^{-\gamma}\chi_\pm(m/\tau^\beta)$ [13] are summarized in Table II.

C. Isothermal magnetocaloric effect, $\lambda > 3$

Before we proceed with the discussion of the peculiarities of the entropic equation of state and of the thermodynamic functions following from it, let us introduce an additional observable—the isothermal magnetocaloric coefficient. It serves as a direct measure of the heat released by the system due to the magnetocaloric effect upon an isothermal increase of the magnetic field and is defined as (see, e.g., [26])

$$M_T = -T\frac{\partial M}{\partial T}\Big|_H . \quad (40)$$

In contradistinction to the heat capacity, which often does not diverge or is a weakly divergent quantity for many $3d$ systems, the magnetocaloric coefficient is frequently strongly divergent at second-order phase transitions [20,26] and therefore it is instructive to analyze how this behavior is modified by a scale-free network. Using Maxwell relations, M_T can be obtained both from the magnetic or from the entropic equations of state, Eqs. (17) and (18). Therefore, an equivalent representation to the one given in (40) reads

$$M_T = -T\frac{\partial S}{\partial H}\Big|_T . \quad (41)$$

Analogous to the first equation in (7), we define the dimensionless isothermal magnetocaloric coefficient as

$$m_T = \frac{M_T}{M_0} . \quad (42)$$

From the above representation, this is

$$m_T = -(1\pm\tau)\frac{\partial s(m,\tau)}{\partial m}\Big|_\tau \frac{\partial m(\tau,h)}{\partial h}\Big|_\tau . \quad (43)$$

Recognizing that the last term in (43) is a dimensionless isothermal susceptibility $\chi_T(\tau,m)$ and writing it in the scaling form

$$\chi_T = \tau^{-\gamma}\chi_\pm(x), \quad x = m/\tau^\beta , \quad (44)$$

we arrive at the scaling representation for the dimensionless isothermal magnetocaloric coefficient m_T,

$$m_T = (1\pm\tau)\tau^{-\omega}\mathcal{M}_\pm(x), \quad (45)$$

with the scaling function

$$\mathcal{M}_\pm(x) = x\chi_\pm(x), \quad (46)$$

and a scaling relation for the isothermal magnetocaloric coefficient critical exponent ω,

$$\omega = 1-\beta . \quad (47)$$

While the equality (47) is a general one and directly follows from the scaling form of Eq. (43), the relation (46) between functions $\mathcal{M}_\pm(x)$ and $\chi_\pm(x)$ holds only for systems where the entropy scaling function has the simple representation (27).

As noticed above, another way to obtain m_T is to start from the magnetic equation of state using the representation (40). Then one obtains

$$m_T = \mp(1\pm\tau)\frac{\partial m(\tau,h)}{\partial \tau}\Big|_h . \quad (48)$$

Comparing this expression with the formulas obtained above for the heat capacity (23) and (24), one arrives at the relation between the scaling functions $\mathcal{M}_\pm(x)$ and $\mathcal{C}_\pm(x)$,

$$\mathcal{C}_\pm(x) = x\mathcal{M}_\pm(x), \quad (49)$$

which, in particular, leads to [cf. (46)]

$$\mathcal{C}_\pm(x) = x^2\chi_\pm(x). \quad (50)$$

TABLE I. Critical exponents governing temperature and field dependencies of the thermodynamic quantities for different values of λ.

	α	β	γ	δ	ω	α_c	γ_c	ω_c
$\lambda \geqslant 5$	0	1/2	1	3	1/2	0	2/3	1/3
$3<\lambda<5$	$\frac{\lambda-5}{\lambda-3}$	$\frac{1}{\lambda-3}$	1	$\lambda-2$	$\frac{\lambda-4}{\lambda-3}$	$\frac{\lambda-5}{\lambda-2}$	$\frac{\lambda-3}{\lambda-2}$	$\frac{\lambda-4}{\lambda-2}$

The scaling function $\mathcal{M}_\pm(x)$ defined above is displayed for different ranges of the values of λ in Table II. For the critical exponents ω, we get

$$\omega = \frac{\lambda-4}{\lambda-3}, \quad 3<\lambda<5; \quad \omega = 1/2, \quad \lambda>5. \quad (51)$$

It is easy to find the scaling relation for the critical exponent ω_c that governs the field dependence of $m_T(\tau=0,h)$ [20],

$$\omega_c = \frac{1-\beta}{\beta\delta} . \quad (52)$$

The values of this exponent read

$$\omega_c = \frac{\lambda-4}{\lambda-2}, \quad 3<\lambda<5; \quad \omega_c = 1/3, \quad \lambda>5. \quad (53)$$

Thus while c_h does not diverge ($\alpha<0$) for the entire range $3<\lambda<5$, m_T is divergent ($\omega>0$) over half that range $4<\lambda<5$, and is a better locator of the phase transition there. The above calculated exponents ω, ω_c are displayed together with other exponents in the comprehensive Table I, which presents a summary of the data concerning the temperature and field behavior of different thermodynamic quantities in the vicinity of the critical point for different values of λ. In the course of the analysis of different types of critical phenomena in scale-free networks [1], it has been revealed that the onset of divergencies of moments of the node-degree distribution function $P(k)$, Eq. (1), relates to changes in the scaling scenario of these systems. As one can see from (51) and (53), the exponents ω and ω_c change their sign to become negative for $\lambda<4$: m_T is no longer divergent at the critical point in the region $3<\lambda<4$. Therefore, along with the two marginal values of $\lambda=5$ and 3, which correspond to the divergencies of $\langle k^4\rangle$ and $\langle k^2\rangle$ and define the "window" of nontrivial critical behavior on a scale-free network, the divergence of the third moment of the node-degree distribution $\langle k^3\rangle$ leads to a qualitative change in the critical behavior of the isothermal magnetocaloric coefficient.

III. DISCUSSION

As one can see from Table I, the heat-capacity exponent α is negative in the region $3<\lambda<5$ where a nontrivial $\alpha(\lambda)$ dependence is observed. This means that the singular part of the heat capacity c_h vanishes at T_c. Taken that c_h vanishes also at $T=0$ and that it is a positive smooth function of T in between, one concludes that it has a maximum at some temperature T_0, where $0<T_0<T_c$ for any $3<\lambda<5$. Therefore, the energy fluctuations are maximal at T_0 (see [12] for more details). Such behavior is a generic feature of systems with $\alpha<0$; other examples include the three-dimensional

TABLE II. Scaling functions and amplitude ratios near the critical point in scale-free networks. The scaling variable is $x = m/\tau^{\beta}$. The ratio Γ^+/Γ^- is taken from Ref. [12] and scaling functions $f_\pm$, $H_\pm$, and $\chi_\pm$ follow from Ref. [13].

	$3 < \lambda < 5$	$\lambda > 5$
$f_\pm$	$\pm x^2/2 + x^{\lambda-1}/4$	$\pm x^2/2 + x^4/4$
$H_\pm$	$\frac{\lambda-1}{4}x^{\lambda-2} \pm x$	$x^3 \pm x$
$\chi_\pm$	$\frac{1}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4 \pm 1}$	$\frac{1}{3x^2 \pm 1}$
$\mathcal{S}$	$-x^2/2$	$-x^2/2$
$\mathcal{C}_\pm$	$\frac{x^2}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4 \pm 1}$	$\frac{x^2}{3x^2 \pm 1}$
$\mathcal{M}_\pm$	$\frac{x}{(\lambda-1)(\lambda-2)x^{\lambda-3}/4 \pm 1}$	$\frac{x}{3x^2 \pm 1}$
A^+/A^-	0	0
Γ^+/Γ^-	$\lambda - 3$	2
R_χ	1	1
R_C	0	0
R_A	$\frac{1}{\lambda-2}[\frac{4}{\lambda-1}]^{\frac{\lambda-5}{(\lambda-2)(\lambda-3)}}$	1/3

Heisenberg and planar magnets [27], liquid helium [28], and disordered uniaxial magnets [29]. From Eqs. (31) and (36), one finds that $c_h(T > T_c, h = 0) = 0$ for any λ. This leads to the following amplitude ratio, which holds for all $\lambda > 3$:

$$A^+/A^- = 0. \tag{54}$$

Amplitude ratios are known to be universal along with the scaling functions and critical exponents (see, e.g., [7]). It is appropriate to adduce here how these ratios change for systems on scale-free networks. The results are summarized in the lower part of Table II. In addition to the heat-capacity amplitude ratio (54), the isothermal magnetic-susceptibility amplitude ratio appears to be λ-dependent for $3 < \lambda < 5$: $\Gamma^+/\Gamma^- = \lambda - 3$ [12]. Using the expressions (28), (31), (34), and (36), it is straightforward to find for the other amplitudes for $\lambda > 5$,

$$B = D_c = 1, \qquad A_c = 1/3, \tag{55}$$

and for $3 < \lambda < 5$,

$$B = \left(\frac{4}{\lambda - 1}\right)^{1/(\lambda-3)}, \qquad D_c = \frac{\lambda - 1}{4},$$
$$A_c = \frac{1}{\lambda - 2}\left(\frac{4}{\lambda - 1}\right)^{3/(\lambda-2)}. \tag{56}$$

Now, defining three more amplitude ratios by [7,16,17]

$$R_\chi = \Gamma^+ D_c B^{\delta-1}, \tag{57}$$
$$R_c = A^+\Gamma^+/B^2, \tag{58}$$
$$R_A = A_c D_c^{-(1+\alpha_c)} B^{-2/\beta}, \tag{59}$$

and substituting into these ratios the amplitudes (55) and (56), we arrive at their values for the scale-free network, as listed in Table II.

Let us concentrate now on the scaling functions. As noted in Sec. II, the entropy scaling function $\mathcal{S}_\pm(x)$, which in the usual Landau theory is given by Eq. (27), keeps its form also in the case of scale-free networks with $3 < \lambda < 5$. However, the constant-magnetic-field heat-capacity scaling function $\mathcal{C}_\pm$ essentially changes in this region. In Fig. 1, we plot $\mathcal{C}_\pm$ as a function of the scaling variable $x = m/\tau^{\beta}$ for different values of λ. The most striking feature in the behavior of the scaling function is that its asymptotics for large x changes for $\lambda < 5$. Indeed, for $\lambda > 5$ the asymptotic value is given by a constant, $\mathcal{C}_\pm(x \to \infty) = 1/3$, whereas in the range $3 < \lambda < 5$ the function behaves as a power law,

$$\mathcal{C}_\pm(x \to \infty) = \frac{4}{(\lambda - 1)(\lambda - 2)} x^{5-\lambda}. \tag{60}$$

In turn, this is reflected in the development of a minimum in the C_- branch of the function as λ decreases (see the figure).

Another particular feature of the plots of Fig. 1 is inherent to the presentation of the scaling plots in the $\mathcal{C}_-$-x plane and is connected to the presence of a pole in $\mathcal{C}_\pm(x)$ for small x. As one sees immediately from Eqs. (31) and (36), this pole occurs at $x = 1/\sqrt{3}$ and $x = \{4/[(\lambda - 1)(\lambda - 2)]\}^{1/(\lambda-3)}$ for $\lambda = 5$ and $3 < \lambda < 5$, correspondingly. However, the physical values of the scaling variable x do not cover the region where the pole occurs. Indeed, from the magnetic equations of state (28) and (34), one may obtain the solutions for the magnetization at zero magnetic field $m(\tau, h = 0)$. Taking that a nonzero magnetic

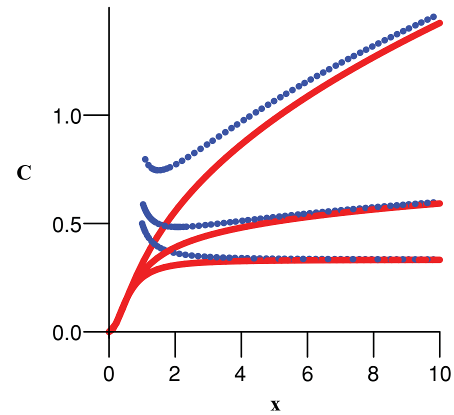

FIG. 1. (Color online) Heat-capacity scaling functions $\mathcal{C}_-(x)$ (dotted blue curves) and $\mathcal{C}_+(x)$ (solid red curves) as functions of the scaling variable $x = m/\tau^{\beta}$ at $\lambda > 5$, $\lambda = 4.8$, and $\lambda = 4.5$ (lower, middle, and upper pairs of curves, respectively).

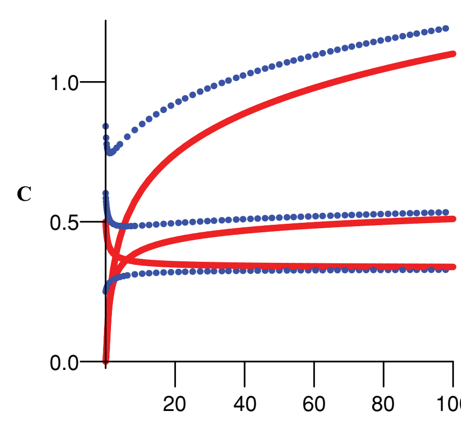

FIG. 2. (Color online) Heat-capacity scaling functions $\mathcal{C}_-(y)$ (dotted blue curves) and $\mathcal{C}_+(y)$ (solid red curves) as functions of the scaling variable $y = h/\tau^{\beta\delta}$ at $\lambda > 5$, $\lambda = 4.8$, and $\lambda = 4.5$ (lower, middle, and upper pairs of curves, respectively).

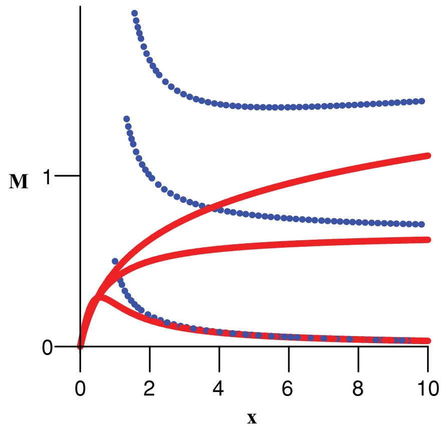

FIG. 3. (Color online) Scaling functions for the isothermal magnetocaloric coefficient. $\mathcal{M}_-(x)$ (dotted blue curves) and $\mathcal{M}_+(x)$ (solid red curves) as functions of the scaling variable $x = m/\tau^{\beta}$ at $\lambda > 5$, $\lambda = 4$, and $\lambda = 3.8$ (lower, middle, and upper pairs of curves, respectively). The decay that is observed for $\lambda > 4$ switches to power-law growth for $\lambda > 4$. The functions approach the constant value $\mathcal{M}_{\pm} = 2/3$ for $\lambda = 4$.

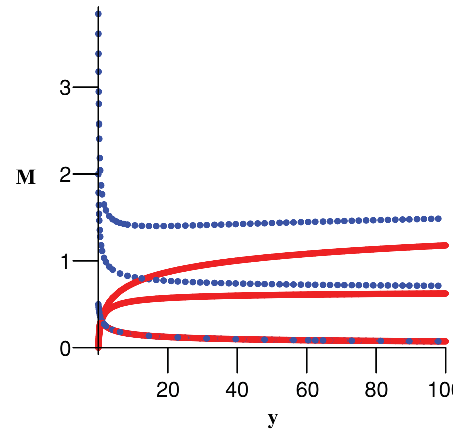

FIG. 4. (Color online) Isothermal magnetocaloric coefficient scaling functions $\mathcal{M}_-(y)$ (dotted blue curves) and $\mathcal{M}_+(y)$ (solid red curves) as functions of the scaling variable $y = m/\tau^{\beta\delta}$ at $\lambda > 5$, $\lambda = 4$, and $\lambda = 3.8$ (lower, middle, and upper pairs of curves, respectively).

field must increase the value of m, one arrives at the following minimal values of the scaling variable x:

$$x_{\min} = 1, \quad \lambda > 5, \tag{61}$$

$$x_{\min} = \left(\frac{4}{\lambda - 1}\right)^{1/(\lambda-3)}, \quad 3 < \lambda < 5. \tag{62}$$

Therefore, the curves for the scaling function $\mathcal{C}_-$ in Fig. 1 originate at the corresponding minimal values of x defined by the relations (61) and (62).

In turn, as explained in Sec. II, one may reexpress the scaling function $\mathcal{C}_{\pm}$ in terms of the scaled magnetic field y using Eq. (37). Corresponding plots for the scaling function $\mathcal{C}_{\pm}(y)$ in this variable are given in Fig. 2 for different values of λ. Again, one observes a change in the asymptotics of the scaling function: instead of a constant at $\lambda > 5$, for $3 < \lambda < 5$ the asymptotic functional dependence is given by

$$\mathcal{C}_{\pm}(y \to \infty) = \frac{1}{(\lambda - 2)}\left(\frac{4}{(\lambda - 1)}\right)^{\frac{3}{\lambda-2}} y^{\frac{5-\lambda}{\lambda-2}}. \tag{63}$$

In Figs. 3 and 4, we give the plots of the scaling functions $\mathcal{M}_{\pm}$ for the isothermal magnetocaloric coefficient in the scaling variables $x = m/\tau^{\beta}$ and $y = h/\tau^{\beta\delta}$, respectively. As discussed at the end of the previous section, m_T changes its behavior at $\lambda = 4$. This feature is also reflected in the behavior of the scaling functions: their asymptotics changes at $\lambda = 4$. Indeed, for $\lambda > 5$ the function decays as $\mathcal{M}_{\pm}(x \to \infty) \sim 1/3x$, whereas from the asymptotic behavior in the region $3 < \lambda < 5$,

$$\mathcal{M}_{\pm}(x \to \infty) = \frac{4}{(\lambda - 1)(\lambda - 2)}\, x^{4-\lambda}, \tag{64}$$

one concludes that for $\lambda < 4$ the power-law decay switches to a power-law growth, while $\mathcal{M}_{\pm}(x \to \infty) = \text{const}$ for the marginal value $\lambda = 4$. The corresponding asymptotics in the variables y is of the form

$$\mathcal{M}_{\pm}(y \to \infty) = \frac{1}{3y^{1/3}}, \quad \lambda > 5,$$
$$\mathcal{M}_{\pm}(y \to \infty) = \frac{1}{(\lambda - 2)}\left(\frac{4}{(\lambda - 1)}\right)^{\frac{2}{\lambda-2}} y^{\frac{4-\lambda}{\lambda-2}}, \quad 3 < \lambda < 5. \tag{65}$$

IV. CONCLUSIONS

Usually the universality of critical phenomena is attributed to the presence of only a few relevant global parameters. Different systems that share the same values of these global variables manifest the same criticality. A classical example is given by the famous 3d Ising model universality class that is inherent to the critical behavior of such differing systems as uniaxial magnets, simple fluids, or binary alloys. The critical behavior in all these systems is quantitatively described by the same values of the critical exponents, amplitude ratios, and by the same form of the scaling functions. As this paper demonstrates, in particular the usual "Euclidean space" understanding of universality of critical phenomena breaks down if the critical behavior occurs on a scale-free network. The presence of high-degree vertices (hubs) may lead to substantial changes in ordering processes. The parameter that controls the "importance" of the hubs is the node-degree distribution exponent λ, Eq. (1), and it is this parameter that plays the role of a global variable as far as the critical behavior is considered.

In particular, the universal quantities that govern criticality become λ-dependent for small enough λ and in this way the network structure is felt. However, the presence of magnetization is necessary to "feel" the network structure. To give an example, the structure matters for $T < T_c$ at any h and for $T > T_c$ for $h \neq 0$ (cf. that amplitude Γ_- is λ-dependent, whereas Γ_+ is not). Another interesting observation is that the fluctuation in network structure only enters via the magnetization, and since the entropy S is measured at constant magnetization, it is given by a usual Landau-like mean-field value for any $\lambda > 3$. This makes a difference between the global parameters λ and dimensionality d regarding the influence of fluctuations on calculating the singular behavior of the physical quantities: no renormalization-group calculation is necessary in this instance.

In this paper, we completed the quantitative description of critical behavior in uncorrelated scale-free networks by calculating the entropic equation of state, the resulting scaling functions, as well as the universal amplitude ratios. The corresponding formulas, together with other data for the critical exponents and amplitudes, are summarized in Tables I and II. They constitute a comprehensive list of observables that describe the scaling and characterize the criticality in scale-free networks.

The starting point for our study was the asymptotic form of the free energy in the vicinity of a critical point in an uncorrelated scale-free network, Eqs. (3) and (4). The validity of this expression has been proven at different levels of rigor using microscopic approaches based on the recursion relations [9], the replica method [8], or phenomenological Landau approaches [10,12] as well as mean-field theory [11]. It is instructive to note here that, because the networks under discussion have a local treelike structure, the mean-field approximation is asymptotically exact. One of the consequences of this fact is that the values of the exponents do not change if an $O(m)$-symmetrical order parameter is considered (see, e.g., [12]), as is usual in the Landau theory. Another consequence is an obvious restriction of the theory developed above to the class of the so-called equilibrium random networks [9]: the undirected graphs, maximally random under the constraint that their degree distribution is a given one, Eq. (1) for the case considered here.

ACKNOWLEDGMENTS

This work was supported by the Austrian Fonds zur Förderung der wissenschaftlichen Forschung (Project No. P19583-N20), the EU Programme FP7-People-2010-IRSES (Project No 269139), and the ARF Scheme of Coventry University. We thank J. F. F. Mendes and a referee for useful comments.

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