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FIELD THEORETICAL APPROACH TO BICRITICAL AND TETRACRITICAL BEHAVIOR: STATICS AND DYNAMICS

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We discuss the static and dynamic multicritical behavior of three-dimensional systems of $O(n_\|)\oplus O(n_\perp)$ symmetry as it is explained by the field theoretical renormalization group method. Whereas the static renormalization group functions are currently known within high order expansions, we show that an account of two loop contributions refined by an appropriate resummation technique gives an accurate quantitative description of the multicritical behavior. One of the essential features of the static multicritical behavior obtained already in two loop order for the interesting case of an antiferromagnet in a magnetic field ($n_\| = 1$, $n_\perp = 2$) is the stability of the biconical fixed point and the neighborhood of the stability border lines to the other fixed points leading to very small transient exponents. We further pursue an analysis of dynamical multicritical behavior choosing different forms of critical dynamics and calculating asymptotic and effective dynamical exponents within the minimal subtraction scheme.

Key words: critical behavior, multicritical points, renormalization group.

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I. INTRODUCTION

Among the milestone contributions of N. N. Bogolyubov that shaped modern theoretical physics one definitely should mention his and D. N. Shirkov's work on the renormalization group (RG) [1]. Three papers on RG written in the mid-fifties by three different groups [2] addressed quantum electrodynamics problems. However, very soon their importance was realized in — at first sight — the very different field of phase transition and critical phenomena. It is generally recognized by now that the success in conceptual understanding and quantitative description of behavior in the vicinity of critical points in different condensed matter systems is due to the effective application of the RG ideas originating from the above papers [3]. It is our pleasure to contribute to these Proceedings[1] by a short review of recent work done by the application of the field theoretical RG approach to the analysis of multicritical phenomena.

Multicritical points appear on phase diagrams of various systems that contain several phase transition lines. In the vicinity of the meeting points of such lines the multicritical behavior is observed, which is characterized by competition of different types of ordering. Prominent examples are given by the antiferromagnets in an external magnetic field like $GdALO_3$, MnF_2, $MnCl_24D_2O$, Mn_2AS_4 (A = Si or Ge) [4]. Other examples are given by the layered cuprate antiferromagnets like $(Ca, La)_{14}Cu_{24}O_{41}$. Schematic phase diagrams of such systems are shown in Fig. 1 in a $H-T$ plane. There, multicritical points of two different types are manifested. At a *bicritical* point (Fig. 1**a**) three phases are in coexistence, whereas four phases coexist in the *tetracritical* point (Fig. 1**b**). On a more general level, the multicritical behavior is inherent to a critical system when some "nonordering" field is applied. Such a field (alongside the magnetic field H this may be pressure, stress, etc.) may alter non-universal parameters of the system and lead to the appearance of lines of phase transition points. Besides the above example that concern the shift of the Néel point of anisotropic antiferromagnets by a uniform magnetic filed, other examples of multicritical behavior are observed at a shift of the Curie points under applied pressure or depression of the λ point in ^4He at dilution by ^3He [5].

A field theoretic description of multicritical behavior starts with a static effective Hamiltonian for an n-component field $\Phi = (\boldsymbol{\phi}_\|, \boldsymbol{\phi}_\perp)$ of $O(n_\|)\oplus O(n_\perp)$ symmetry ($n_\| + n_\perp = n$). An account of the interaction between the two order parameters $\boldsymbol{\phi}_\|$ and $\boldsymbol{\phi}_\perp$ leads to different types of multicritical behavior connected with the stable fixed point (FP) found in the RG treatment [6–11, 14]. In particular, the bicritical point (Fig. 1**a**) has been connected with the stability of the *isotropic Heisenberg* fixed

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point of $O(n_\| + n_\perp)$ symmetry, whereas the tetracritical point (Fig. 1**b**) corresponds to a FP of $O(n_\|) \oplus O(n_\perp)$ symmetry, which might be either the so called *biconical* FP or the *decoupling* FP. In the last FP the parallel and the perpendicular components of the order parameter are asymptotically decoupled. If no FP is reached the multicritical point might be of the first order, i. e. a triple point.

Quite recently the possible types of phase diagrams in the $H-T$ plane of three dimensional uniaxial anisotropic antiferromagnets have been studied by Monte Carlo simulations [12]. For $n_\| = 1$ and $n_\perp = 2$ a phase diagram with a bicritical point has been found in agreement with earlier simulations [13], but contrary to the results of RG theory in higher loop orders [11].

The dynamics of antiferromagnets in a magnetic field is quite complicated. To account for conservation laws present in such systems, the dynamical equations of motion should contain coupling terms between the two order parameters (the components of the staggered magnetization parallel and perpendicular to the magnetic field, $\boldsymbol{\phi}_\|$ and $\boldsymbol{\phi}_\perp$) and conserved densities (e. g. the parallel component of the magnetization or energy density). The first formulation of the equations of motion at multicritical points has been done in Ref. [17]. The simplest form of dynamics assumes the relaxational behavior for the two order parameters $\boldsymbol{\phi}_\|$ and $\boldsymbol{\phi}_\perp$ (the so-called model A) [15, 18]. The dynamical multicritical behavior within the one-loop approximation has been considered in [17] on the basis of the static one loop results [9]. A further step to the complete model is to include the diffusive dynamics of the slow conserved density leading to a model C like extension. This model has been studied in one loop order in Refs. [17, 19, 20] taking into account only a part of dynamical two loop order terms and one loop statics. In order to get more insight in the dynamics in the vicinity of multicritical points, recently we have reconsidered the above dynamical models within the two loop approximation [16, 21].

In what follows below we briefly summarize an outcome of an RG analysis of the multicritical behavior paying special attention to an impact of the non-universal contributions to an asymptotic behavior. In particular, we will show that an account of two loop part of the RG expansions refined by an appropriate resummation technique gives an accurate quantitative description of the static multicritical behavior. Furthermore, we pursue an analysis of dynamical multicritical behavior choosing different forms of critical dynamics and calculating asymptotic and effective dynamical exponents.

II. RG FLOWS AND STATIC MULTICRITICAL BEHAVIOR

The generalized static $O(n_\|) \oplus O(n_\perp)$-symmetrical effective Hamiltonian that results from the decomposition of the n-component order parameter field into two mutually interacting fields $\boldsymbol{\phi}_\|$ and $\boldsymbol{\phi}_\perp$ of different irreducible representations of dimensions $n_\|$ and $n_\perp$, $n = n_\| + n_\perp$, reads:

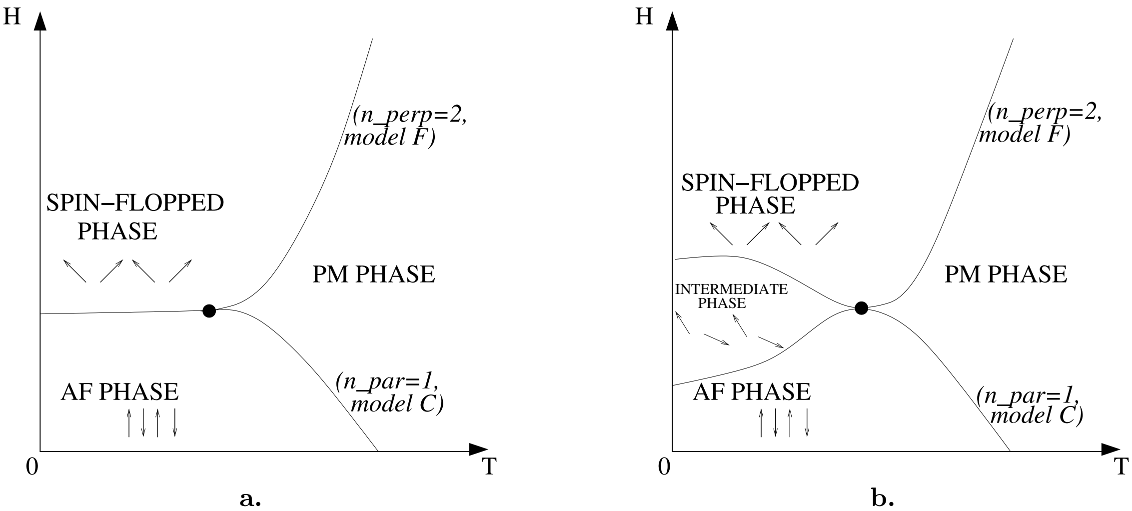


Fig. 1. Typical phase diagrams of anisotropic antiferromagnets in a uniform parallel external magnetic field H. Types of ordering are schematically shown by arrows. **a**: the *bicritical* point. Three phases — an antiferromagnetic (AF) phase, a spin flop phase and the paramagnetic (PM) phase are in coexistence. The phase transition lines to the paramagnetic phase are second order transition lines, whereas the transition line between the spin flop and the antiferromagnetic phase is of the first order. **b**: the *tetracritical* point. Four phases — an antiferromagnetic phase, a spin flop phase, an intermediate or mixed phase and the paramagnetic phase — are in coexistence. All transition lines are of the second order in this case. Also indicated is the dynamical universality class of the transition from the paramagnetic to the corresponding ordered phase according to the classification of Hohenberg and Halperin [15] for the three component antiferromagnet.

$$\mathcal{H}=\int d^dx\Bigg\{\frac{1}{2}\mathring{r}_\perp\boldsymbol{\phi}_{\perp0}\cdot\boldsymbol{\phi}_{\perp0}+\frac{1}{2}\sum_{i=1}^{d}\nabla_i\boldsymbol{\phi}_{\perp0}\cdot\nabla_i\boldsymbol{\phi}_{\perp0}+\frac{1}{2}\mathring{r}_\|\boldsymbol{\phi}_{\|0}\cdot\boldsymbol{\phi}_{\|0}+\frac{1}{2}\sum_{i=1}^{d}\nabla_i\boldsymbol{\phi}_{\|0}\cdot\nabla_i\boldsymbol{\phi}_{\|0}$$
$$+\frac{\mathring{u}_\perp}{4!}\Big(\boldsymbol{\phi}_{\perp0}\cdot\boldsymbol{\phi}_{\perp0}\Big)^2+\frac{\mathring{u}_\|}{4!}\Big(\boldsymbol{\phi}_{\|0}\cdot\boldsymbol{\phi}_{\|0}\Big)^2+\frac{2\mathring{u}_\times}{4!}\Big(\boldsymbol{\phi}_{\perp0}\cdot\boldsymbol{\phi}_{\perp0}\Big)\Big(\boldsymbol{\phi}_{\|0}\cdot\boldsymbol{\phi}_{\|0}\Big)\Bigg\}. \quad (1)$$

Here, $\{\mathring{u}_\perp,\mathring{u}_\times,\mathring{u}_\|\}=\{\mathring{u}\}$ and $\mathring{r}_\perp$, $\mathring{r}_\|$ are couplings and masses, correspondingly, index 0 refers to the bare quantities, and central dots stand for scalar products. The decomposition in parallel and perpendicular order parameter components allows to describe the multicritical behavior at the meeting point of two critical lines: (i) the line where $\mathring{r}_\perp$ becomes zero and the $n_\perp$-dimensional components $\boldsymbol{\phi}_{\perp0}$ are the order parameter, and (ii) the line where $\mathring{r}_\|$ becomes zero and the order parameter is $\boldsymbol{\phi}_{\|0}$. At the meeting point both quadratic terms become zero and both components of $\boldsymbol{\phi}_0$ have to be taken into account. As has been predicted already by the one-loop RG analysis [8,9], an effective Hamiltonian (1) describes three different types of multicritical behavior that are governed by three different FPs: (i) the isotropic n component Heisenberg FP, called below $\mathcal{H}(n)$, all fourth order couplings are equal in this FP, (ii) the decoupling FP point $\mathcal{D}$, which consists of a combination of the FPs $\mathcal{H}(n_\perp)$ and $\mathcal{H}(n_\|)$ of two decoupled systems and (iii) the biconical FP, $\mathcal{B}$, with nontrivial nonzero couplings. As revealed by subsequent calculations [10,11] the FP picture does not change qualitatively with an account of higher orders of the perturbation theory. However, the one-loop results attain essential quantitative changes that lead to a drastic modification of the type of a phase diagram. A typical example may be given by the behavior of the β-functions that describe the flow of the fourth order couplings $\{\mathring{u}\}$ under renormalization. The above functions, calculated in the two-loop approximation with the minimal subtraction RG scheme read [14]:

$$\beta_{u_\perp}=-\varepsilon u_\perp+\frac{(n_\perp+8)}{6}u_\perp^2+\frac{n_\|}{6}u_\times^2-\frac{(3n_\perp+14)}{12}u_\perp^3-\frac{5n_\|}{36}u_\perp u_\times^2-\frac{n_\|}{9}u_\times^3, \quad (2)$$

$$\beta_{u_\times}=-\varepsilon u_\times+\frac{(n_\perp+2)}{6}u_\perp u_\times+\frac{(n_\|+2)}{6}u_\times u_\|+\frac{2}{3}u_\times^2-\frac{(n_\perp+n_\|+16)}{72}u_\times^3$$
$$-\frac{(n_\perp+2)}{6}u_\times^2u_\perp-\frac{(n_\|+2)}{6}u_\times^2u_\|-\frac{5(n_\perp+2)}{72}u_\perp^2u_\times-\frac{5(n_\|+2)}{72}u_\times u_\|^2, \quad (3)$$

$$\beta_{u_\|}=-\varepsilon u_\|+\frac{(n_\|+8)}{6}u_\|^2+\frac{n_\perp}{6}u_\times^2-\frac{(3n_\|+14)}{12}u_\|^3-\frac{5n_\perp}{36}u_\| u_\times^2-\frac{n_\perp}{9}u_\times^3. \quad (4)$$

Here, $\{u_\perp,u_\times,u_\|\}=\{u\}$ are renormalized couplings and the space dimension d enters the β-functions via parameter $\varepsilon=4-d$. With the β-functions at hand, one can analyze the flow equations of the fourth-order couplings $\{u\}$:

$$\ell\frac{du_a}{d\ell}=\beta_{u_a}(\{u\}), \quad (5)$$

with $a=\perp,\ \|,\ \times$ and the flow parameter ℓ, and find the FPs $\{u^*\}$ of these equations as the solutions of the system of equations

$$\beta_{u_a}(\{u^*\})=0. \quad (6)$$

Which of these FPs is the stable one depends on the number of components $n_\perp$ and $n_\|$ and the dimension d of space. The scaling properties depend on the symmetry of the stable FP.

There are two alternative ways to look for the solutions of the FP equations (6) and, subsequently, for the scaling properties of the system. In one approach, the ε-expansion, the solutions are obtained as a series in ε and then evaluated at the value of interest (at $\varepsilon=1$ for $d=3$ theories). Alternatively, one may solve a system of non-linear equations directly at the dimensionality of space of interest (e. g. at $\varepsilon=1$) [22] and obtain the FP coordinates numerically. The RG expansions being divergent [23], special resummation techniques are used to get convergent results [24]. As we have discussed already above, depending on the values of $n_\|$, $n_\perp$, and d, the multicritical behavior is governed by one of the three non-trivial FPs: $\mathcal{H}$ $\{u_\perp^*=u_\times^*=u_\|^*\}$, $\mathcal{B}$ $\{u_\perp^*\neq u_\times^*\neq u_\|^*\}$, and $\mathcal{D}$ $\{u_\perp^*\neq 0, u_\times^*=0, u_\|^*\neq 0\}$. In Fig. 2 we show how the stability of these FPs changes with $n_\|$, $n_\perp$, for $d=3$. There, we compare the first order ε-expansion results [8,9] with the two-loop results [14] obtained within the fixed $d=3$ technique [22]. The two-loop results were obtained applying Padé–Borel resummation technique to functions (2)–(4) [25]. One sees that the borderlines of the FPs stability are drastically shifted to smaller values of order parameter components. Thus, in the case $n_| = 1$ and $n_\perp=2$ FP $\mathcal{B}$ (connected with tetracriticality) is stable in two-loop order contrary to the one-loop calculations where the FP $\mathcal{H}$ (connected with bicriticality) is stable. The resummed higher orders of the perturbation theory do not change this result and do not lead to essential changes in the critical exponents either [11].

As usual, the asymptotic values of the critical exponents are defined by the stable FP values of the corresponding RG ζ-functions, which we do not expose here. Note that in general, there are distinct exponents $\eta_\|$, $\eta_\perp$ governing spacial decay of the order parameter correlations in directions parallel and perpendicular to the anisotropy axis. As a consequence, there is a pair of γ-exponents, $\gamma_\parallel$, $\gamma_\perp$ that govern corresponding isothermal magnetic susceptibilities. However, the above RG procedure assumes that the multicritical system is described by a single diverging length scale and therefore by one correlation length ξ and one corresponding critical exponent ν. This does not hold for decoupled systems where two length scales are present and the usual scaling laws with one length scale break down [9]. We give typical numerical values of the exponents in Table 1.

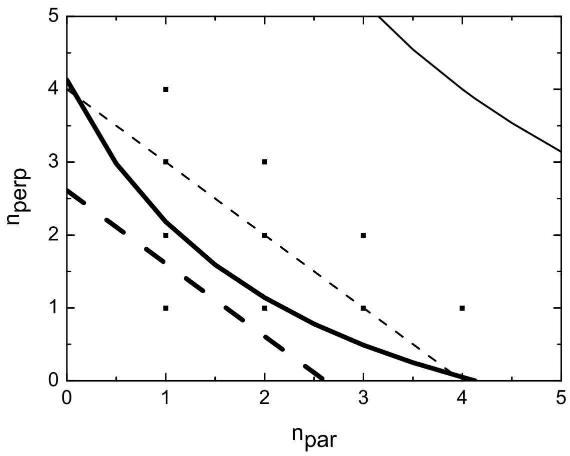


Fig. 2. Regions of FPs stability in the in the $n_\parallel - n_\perp$-plane, $d = 3$. The lines separate regions where Heisenberg FP $\mathcal{H}$, biconical FP $\mathcal{B}$ and decoupling FP $\mathcal{D}$ are stable (from left to right). Shown are the $\mathcal{HB}$-stability borderlines (dashed lines) and $\mathcal{BD}$-stability borderlines (solid lines), in one loop order (thin lines) and resummed two loop order (thick lines). The dots indicate low integer values for order parameter components [26].

Whereas the asymptotic critical exponent values are determined *strictly at* the FP and correspond to the scaling behavior at the multicritical point, of special interest are the effective critical exponents which are observed *in the vicinity* of the multicritical point. These are the effective exponents that often are observed experimentally and are measured in MC simulations. In the RG framework, one may estimate the effective exponents from the values of corresponding RG ζ-functions calculated along the RG flow and relate the flow parameter ℓ to the distance to the multicritical point. In Fig. 3 we show the resummed [25] RG flow of Eqs. (5) for different initial conditions [14]. The unstable FPs are shown as filled spheres, the stable biconical FP as a filled cube. Let us note that the neighborhood of the stability border lines to the other FPs leads to very small transient exponents. Therefore, the stable FP is not reached for the value of the flow parameter chosen in Fig. 3 (there, the flow parameter has been changed in the interval $-40 \leq \ln \ell \leq 0$).

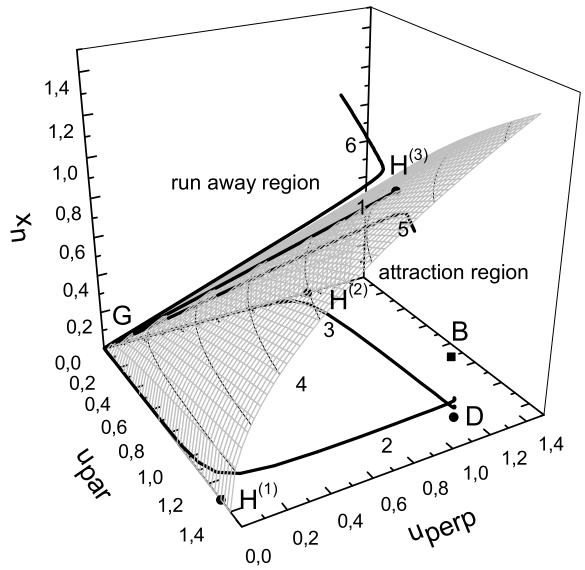
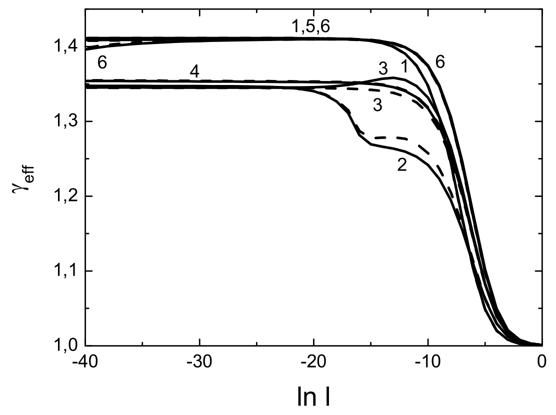


Fig. 3. Resummed RG flow of Eqs. (5) for different initial conditions at $d = 3$, $n_\parallel = 1$, $n_\perp = 2$. The unstable FPs are shown as a filled spheres, the stable biconical FP as filled cube. The FPs points are connected by separatrices defining the surface which encloses the attraction region [26].

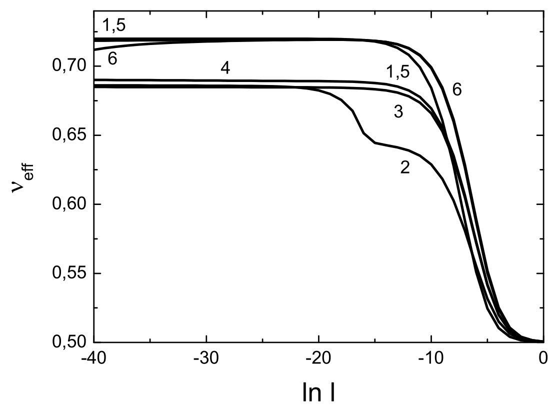
Defining the effective exponents as explained above, one can evaluate their numerical values along the RG flows of Fig. 3 and in this way predict possible outcome of measuring the scaling properties of different observables at the multicritical point. As two typical examples, we show in Fig. 4 the change of values of isothermal susceptibility effective exponents $\gamma_\parallel$, $\gamma_\perp$ and of the correlation length critical exponent ν as the multicritical point is being approached, the limit $T \to T_c$ corresponds to the limit $\ell \to 0$.

Reference	FP	$\eta_\perp$	$\eta_\parallel$	$\gamma_\perp$	$\gamma_\parallel$	ν
[14]	$\mathcal{B}$	0.037	0.037	1.366	1.366	0.696
[14]	$\mathcal{H}(3)$	0.040	0.040	1.411	1.411	0.720
[6]	$\mathcal{B}$	0	0	1.222	1.222	0.611
[6]	$\mathcal{H}(3)$	0	0	1.227	1.227	0.611
[11]	$\mathcal{B}$	0.037(5)	0.037(5)	*1.37(7)*	*1.37(7)*	0.70(3)
[27]	$\mathcal{H}(3)$	0.0375(45)	0.0375(45)	1.382(9)	1.382(9)	0.7045(55)

Table 1. Critical exponents of the $O(1) \oplus O(2)$ model obtained in different approximations. [14]: resummation of the two-loop RG series at fixed $d = 3$; [6]: first order ε-expansion, [11,27]: resummed fifth order ε-expansion. Numbers, shown in italic were obtained via familiar scaling relations.



a.



b.

Fig. 4. Effective exponents of different observables in the vicinity of a multicritical point for the the flows of Fig. 3. **a**: isothermal magnetic susceptibility (solid curves: $\gamma_{\|}$, dashed curves: $\gamma_{\perp}$); **b**: correlation length [26].

Before passing to a discussion of some peculiarities of the dynamic multicritical behavior, let us note a particular feature of the $O(n_{\|}) \oplus O(n_{\perp})$ model that becomes evident from the above analysis of the statics. As the stability analysis shows, for the physically interesting case $d = 3$, $n_{\|} = 1$, $n_{\perp} = 2$ the asymptotic behavior is governed by the biconical FP $\mathcal{B}$. Therefore, the tetracritical point is realized (cf. Fig. 1**a**). However, depending on the particular microscopic non-universal characteristics of a given system, one may expect a variety of different scenarios for the multicritical behavior, including the triple point (that corresponds to the run away solutions of the RG flow equations, cf. Fig. 3) and bicritical point (when for certain initial condition the Heisenberg FP $\mathcal{H}$ is reached).

III. DYNAMICS IN THE VICINITY OF MULTICRITICAL POINTS

The above-sketched particular features of the static multicritical behavior are further manifested if the critical dynamics is addressed. Below, we briefly analyze three different forms of dynamical behavior in the vicinity of multicritical points.

A. Relaxational dynamics (model A)

Let us start from the simplest dynamical model, model A, when one assumes relaxational behavior for the two order parameters $\boldsymbol{\phi}_{\|}$ and $\boldsymbol{\phi}_{\perp}$. This model has been studied in the one-loop approximation in [17], the two-loop results have been obtained in [14]. The model A type Langevin equations of motion describe two order parameters that relax to equilibrium with the relaxation rates (kinetic coefficients) $\mathring{\Gamma}_{\perp}$ and $\mathring{\Gamma}_{\|}$:

$$\frac{\partial \boldsymbol{\phi}_{\perp 0}}{\partial t} = -\mathring{\Gamma}_{\perp} \frac{\delta \mathcal{H}}{\delta \boldsymbol{\phi}_{\perp 0}} + \boldsymbol{\theta}_{\phi_{\perp}}, \tag{7}$$

$$\frac{\partial \boldsymbol{\phi}_{\| 0}}{\partial t} = -\mathring{\Gamma}_{\|} \frac{\delta \mathcal{H}}{\delta \boldsymbol{\phi}_{\| 0}} + \boldsymbol{\theta}_{\phi_{\|}}. \tag{8}$$

Here, $\mathcal{H}$ is the static effective Hamiltonian (1), index 0 refers to bare (unrenormalized) quantities and the stochastic forces $\boldsymbol{\theta}_{\phi_{\perp}}$, $\boldsymbol{\theta}_{\phi_{\|}}$ fulfill Einstein relations

$$\langle \theta^{\alpha}_{\phi_{\perp}}(x,t)\ \theta^{\beta}_{\phi_{\perp}}(x',t')\rangle = 2\mathring{\Gamma}_{\perp}\delta(x-x')\delta(t-t')\delta^{\alpha\beta}, \tag{9}$$

$$\langle \theta^{i}_{\phi_{\|}}(x,t)\ \theta^{j}_{\phi_{\|}}(x',t')\rangle = 2\mathring{\Gamma}_{\|}\delta(x-x')\delta(t-t')\delta^{ij}, \tag{10}$$

with indices $\alpha, \beta = 1, \ldots, n_{\perp}$ and $i, j = 1, \ldots, n_{\|}$ corresponding to the two subspaces.

Application of the RG procedure to study dynamical multicritical behavior relies on the Bausch–Janssen–Wagner approach [28], where the appropriate Lagrangian of the model is studied and dynamic vertex functions are calculated in perturbation theory and renormalized. In such a technique, an essential simplification of calculations is achieved due to the possibility to single out a static part of every dynamic vertex function [29, 30]. Renormalization of the kinetic coefficients gives rise to appropriate β-functions. Here, we reveal the two-loop β-function for the time-scale ratio $v = \Gamma_{\|}/\Gamma_{\perp}$ between the renormalized kinetic coefficients $\Gamma_{\|}$ and $\Gamma_{\perp}$. The function reads [16]:

$$\beta_v = \frac{v}{72}\Bigg\{\Big[(n_{\|}+2)u_{\|}^2 - (n_{\perp}+2)u_{\perp}^2\Big]\left(6\ln\frac{4}{3}-1\right) - n_{\|}\, u_{\times}^2\left[\frac{4}{v}\ln\frac{2(1+v)}{2+v} + 2\ln\frac{(1+v)^2}{v(2+v)} - 1\right] + n_{\perp}\, u_{\times}^2\left[4v\ln\frac{2(1+v)}{1+2v} + 2\ln\frac{(1+v)^2}{1+2v} - 1\right]\Bigg\}. \tag{11}$$

As we have noted in the preceding section discussing the static critical behavior, a non universal effective critical behavior may be observed if the values of the static couplings and the time scale ratio are not in a FP but rather are described by the flow equations. For v the flow equation reads

$$\ell\frac{dv}{d\ell} = \beta_v\big(u_\|(\ell), u_\perp(\ell), u_\times(\ell), v(\ell)\big). \qquad (12)$$

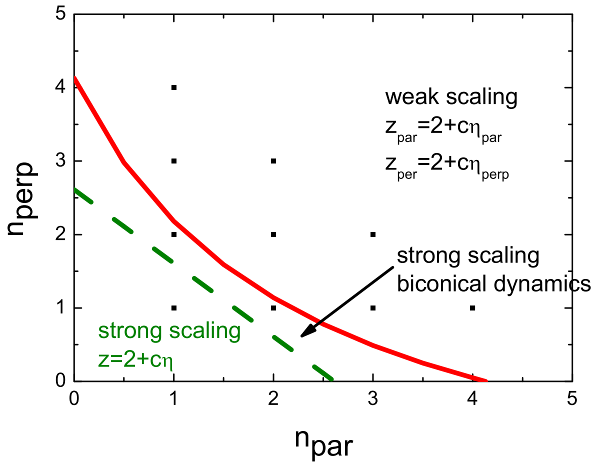


Fig. 5. Regions of different types of the dynamic scaling behavior, $\epsilon = 4 - d = 1$. The rest of notations are as in Fig. 2 [31].

Below we will show some results about non-universal dynamic multicritical behavior obtained with two-loop accuracy. The numerical results for the static part of the RG function were obtained by means of the resummation technique [25], whereas no resummation has been applied to the dynamic functions [16].

One of the quantities of interest that characterize dynamic critical phenomena is the autocorrelation time τ. It is known to diverge as the critical point T_c is approached, the divergency is described by the power law:

$$\tau \sim |T - Tc|^{-\nu z}, \qquad (13)$$

with the universal correlation length and dynamic critical exponents ν and z, correspondingly. In the multicritical phenomena we consider, one distinguishes two dynamical critical exponents, $z_\|$ and $z_\perp$, that govern the power law increase of the autocorrelation time for the order parameters $\boldsymbol{\phi}_\|$ and $\boldsymbol{\phi}_\perp$, correspondingly. In asymptotics they are defined by the stable FP values of the corresponding RG functions. At the strong scaling FP there is only one dynamic time scale and the two exponents are equal whereas at the weak scaling FP they are different and define for each component, parallel and perpendicular, the time scale. As follows from our calculations [16] and as one may see from Fig. 5, the region of stability of the biconical FP $\mathcal{B}$ (physically important case $d = 3$, $n_\| = 1$, $n_\perp = 2$ belongs to this region) is characterized by the strong scaling dynamics: the time relaxation of both order parameters, $\boldsymbol{\phi}_\|$ and $\boldsymbol{\phi}_\perp$ is governed by the same exponent. In Fig. 6 we show an evolution of this exponent $z_{\rm eff}$ to its asymptotic value $z = 2.05$ when the time-scale ratio v is set to its FP value and the static couplings u change along the RG flows of Fig. 3. Since the exponents have not reached their (equal) asymptotic values differences between the parallel and perpendicular components of the order parameter remain.

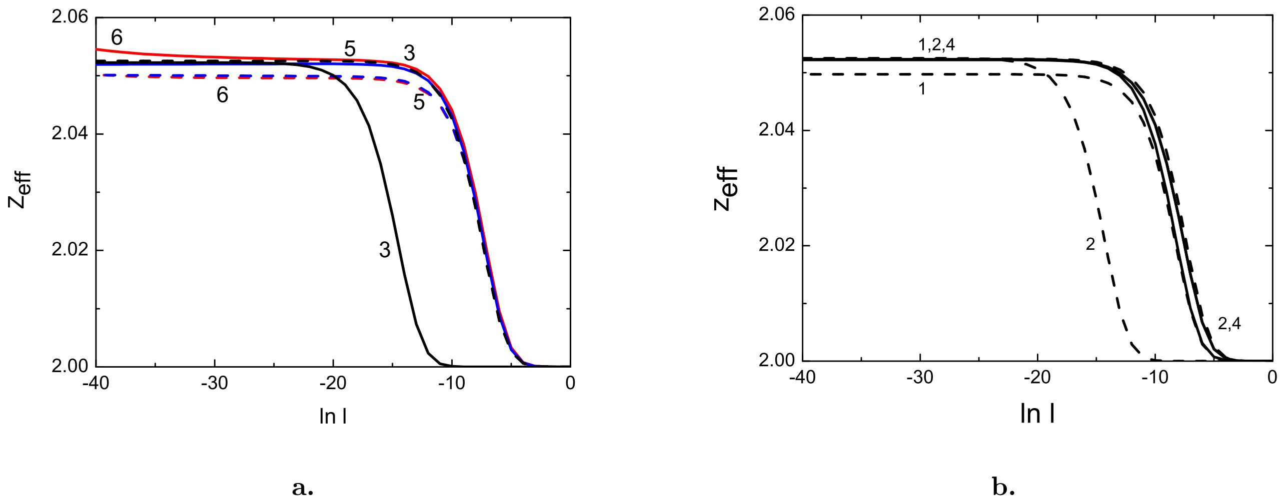
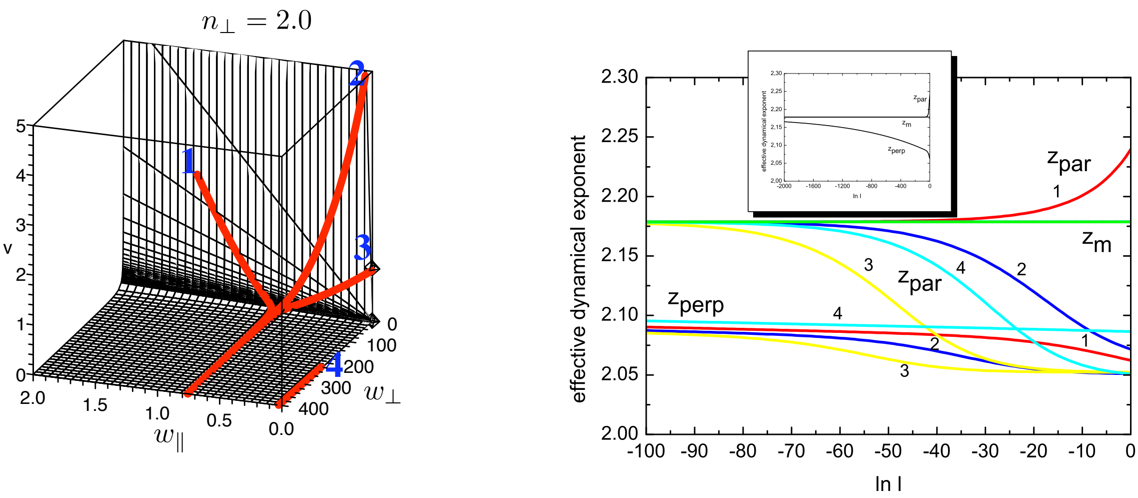


Fig. 6. Model A multicritical dynamics. The effective dynamical exponent for different RG flows in the vicinity of a multicritical point at $d = 3$, $n_\| = 1$, $n_\perp = 2$. The labeling of the flows corresponds to Fig. 3. The exponents for the perpendicular (dashed curves) and parallel (solid curves) components of the order parameter differ in the non asymptotic region [31].



a. **b.**

Fig. 7. Model C effective dynamical multicritical behavior at $d = 3$, $n_\parallel = 1$, $n_\perp = 2$. **a**: dynamical RG flow for different initial conditions numbered from 1 to 4. **b**: effective dynamical exponents $z_\parallel$, $z_\perp$, and z_m calculated along the RG flows of Fig. **7a**, as indicated by the numbers [32].

B. Conservation of magnetization (model C)

A step towards making the description of dynamic phenomena in the vicinity of a multicritical point more realistic is to take into account possible couplings between the order parameters and conserved densities, that is to consider the model C dynamics [15, 18]. In the problem under consideration, there are two types of conserved densities: one is magnetization-like (more precisely, it is the parallel component of the magnetization), another is the energy density. We will not consider this second density here, as far as up to the two-loop order the specific heat critical exponent α is negative for the case $d = 3$, $n_\parallel = 1$, $n_\perp = 2$ which is of most interest here. Therefore, a coupling to the energy density is irrelevant in the RG sense — it vanishes at the FP [30]. An account of both the order parameter and the (conserved) scalar density is achieved by an extension of the static functional (1). Now, the corresponding model C static functional reads:

$$\mathcal{H}^{(C)} = \mathcal{H} + \int d^dx \left(\frac{1}{2} m_0^2 + \frac{1}{2} \mathring{\gamma}_\perp m_0 \boldsymbol{\phi}_{\perp 0} \cdot \boldsymbol{\phi}_{\perp 0} + \frac{1}{2} \mathring{\gamma}_\parallel m_0 \boldsymbol{\phi}_{\parallel 0} \cdot \boldsymbol{\phi}_{\parallel 0} - \mathring{h} m_0 \right). \tag{14}$$

Here, the first term in the right hand side is given by Eq. (1), the density $m_0 \equiv m_0(x)$ is a scalar quantity, $\mathring{h}$ is a field conjugated to m_0, $\mathring{\gamma}_\perp$ and $\mathring{\gamma}_\parallel$ are asymmetric static couplings between the corresponding order parameters and the conserved density.

In their turn, the relaxational equations of motion (7),(8) are now extended by including a diffusion equation for the scalar density:

$$\frac{\partial \boldsymbol{\phi}_{\perp 0}}{\partial t} = -\mathring{\Gamma}_\perp \frac{\delta \mathcal{H}^{(C)}}{\delta \boldsymbol{\phi}_{\perp 0}} + \boldsymbol{\theta}_{\phi_\perp} \,, \tag{15}$$

$$\frac{\partial \boldsymbol{\phi}_{\parallel 0}}{\partial t} = -\mathring{\Gamma}_\parallel \frac{\delta \mathcal{H}^{(C)}}{\delta \boldsymbol{\phi}_{\parallel 0}} + \boldsymbol{\theta}_{\phi_\parallel} \,, \tag{16}$$

$$\frac{\partial m_0}{\partial t} = \mathring{\lambda} \nabla^2 \frac{\delta \mathcal{H}^{(C)}}{\delta m_0} + \theta_m \,. \tag{17}$$

Here, the static functional $\mathcal{H}^{(C)}$ is given by (14), $\mathring{\lambda}$ is a kinetic coefficient of a diffusive type for the scalar density, the rest of notations is as in (7),(8). The stochastic forces $\boldsymbol{\theta}_{\phi_\perp}$, $\boldsymbol{\theta}_{\phi_\parallel}$ satisfy the Einstein relations (9), (10), with an additional Einstein relation for the new stochastic force θ_m:

$$\langle \theta_m(x,t)\; \theta_m(x',t') \rangle = -2\mathring{\lambda} \nabla^2 \delta(x-x') \delta(t-t'). \tag{18}$$

The renormalization of the above introduced asymmetric couplings $\mathring{\gamma}_\perp$, $\mathring{\gamma}_\parallel$ and kinetic coefficient $\mathring{\lambda}$ leads to new RG functions. In particular the RG flow of the time scale ratios

$$w_\perp = \frac{\Gamma_\perp}{\lambda} \,, \qquad w_\parallel = \frac{\Gamma_\parallel}{\lambda} \tag{19}$$

is now governed by the appropriate functions $\beta_{w_\perp}$ and $\beta_{w_\parallel}$, correspondingly. Note that defined for model A time scale ratio v is equally well defined in terms of (19):

$$v \equiv \frac{\Gamma_\parallel}{\Gamma_\perp} = \frac{w_\parallel}{w_\perp} \,. \tag{20}$$

Therefore, the dynamical FP equations:

$$\beta_{w_\perp}(w_\perp^*, w_\parallel^*, v^*) = \beta_{w_\parallel}(w_\perp^*, w_\parallel^*, v^*) = \beta_v(w_\perp^*, w_\parallel^*, v^*) = 0 \tag{21}$$

are now not independent: one of these equations can be eliminated by the relation (20).

Equations of motion (15)–(17) describe time evolution of three different observables. Each of them has its own autocorrelation time which, as the multicritical point is reached, may be governed by an independent dynamical critical exponent. In addition to the two exponents defined in the former subsection, $z_\|$ and $z_\perp$, the dynamical critical exponent z_m for the scalar density is to be considered. Similarly to, as in the model A case, these three exponents may coincide, in the strong scaling dynamical FP or they may differ, in the weak scaling dynamical FP. Complete stability analysis of the model C RG equations in two-loop approximation is given in Ref. [16]. In particular, it is shown that for the case $d=3$, $n_\|=1$, $n_\perp=2$, where the static FP is the biconical FP $\mathcal{B}$, the strong scaling dynamical FP is stable. Physically this means that in asymptotics the multicritical dynamics is characterized by one time scale, and three dynamical exponents coincide. In particular, their asymptotical value was found to be $z_\|=z_\perp=z_m=2.18$ [14]. However, as was revealed in the former sections, the effective multicritical behavior is much richer. In particular, in Fig. 7**a** we show the RG flows calculated for different dynamical initial conditions when the static couplings are chosen to be fixed at their biconical FP values. The stable dynamical (strong scaling) FP lies outside the region shown. Also shown is the surface $v=w_\|/w_\perp$ to which the flow is restricted by condition (20). The RG flows of Fig. 7**a** give rise to a difference in the effective dynamical critical exponents, as shown in Fig. 7**b**. The insert of the figure shows that even for flow parameters as small as $\ln\ell=-2000$ the effective exponent $z_\perp$ has not reached its asymptotic value 2.18.

C. The complete dynamic model (model G)

We now restrict ourselves to the case of $n_\|=1, n_\perp=2$ and include mode coupling terms, which correspond to Larmor terms describing the precession of the alternating magnetization and the magnetization around each other. They are well known from the isotropic antiferromagnet without an external field [33]. Then within an external magnetic field the corresponding equations read

$$\frac{\partial\phi^\alpha_{\perp 0}}{\partial t}=-\mathring{\Gamma}'_\perp\frac{\delta\mathcal{H}^{(C)}}{\delta\phi^\beta_{\perp 0}}+\mathring{\Gamma}''_\perp\epsilon^{\alpha\beta z}\frac{\delta\mathcal{H}^{(C)}}{\delta\phi^\alpha_{\perp 0}} +\mathring{g}\,\epsilon^{\alpha\beta z}\phi^\beta_{\perp 0}\frac{\delta\mathcal{H}^{(C)}}{\delta m_0}+\theta^\alpha_{\phi_\perp}, \quad (22)$$

$$\frac{\partial\phi_{\|0}}{\partial t}=-\mathring{\Gamma}_\|\frac{\delta\mathcal{H}^{(C)}}{\delta\phi_{\|0}}+\theta_{\phi_\|}, \quad (23)$$

$$\frac{\partial m_0}{\partial t}=\mathring{\lambda}\nabla^2\frac{\delta\mathcal{H}^{(C)}}{\delta m_0}+\mathring{g}\,\epsilon^{z\alpha\beta}\phi^\alpha_{\perp 0}\frac{\delta\mathcal{H}^{(C)}}{\delta\phi^\beta_{\perp 0}}+\theta_m. \quad (24)$$

Now α and β indicate the planar components x,y and the Levi-Civita tensor $\varepsilon^{\alpha\beta z}$ with the third index fixed to z has been introduced. The parallel component of the order parameter is its z-component. This component remains just relaxing, whereas the planar components of the order parameter are coupled to the z-component of the magnetization by the precession terms.

A new feature arises because of the simultaneous presence of the mode coupling $\mathring{g}$ and the asymmetric static couplings $\mathring{\gamma}_\perp$ and $\mathring{\gamma}_\|$ in $\mathcal{H}^{(C)}$ (14). The perpendicular relaxation coefficient $\mathring{\Gamma}_\perp$ has to be considered a complex quantity where the imaginary part constitutes a precession term (second term on the right hand side of (22)). Even if in the background such terms are absent they are produced by the renormalization procedure.

The stochastic forces $\boldsymbol{\theta}_{\phi_\perp}$, $\theta_{\phi_\|}$ and θ_m fulfill Einstein relations

$$\langle\theta^\alpha_{\phi_\perp}(x,t)\;\theta^\beta_{\phi_\perp}(x',t')\rangle=2\mathring{\Gamma}'_\perp\delta(x-x')\delta(t-t')\delta^{\alpha\beta}, \quad (25)$$

$$\langle\theta_{\phi_\|}(x,t)\;\theta_{\phi_\|}(x',t')\rangle=2\mathring{\Gamma}_\|\delta(x-x')\delta(t-t'), \quad (26)$$

$$\langle\theta_m(x,t)\;\theta_m(x',t')\rangle=-2\mathring{\lambda}\nabla^2\delta(x-x')\delta(t-t'). \quad (27)$$

This model has been solved in one loop order in [17] using the one loop results of statics. As has been already seen for the simpler dynamics models changes are expected in two loop order both by the statics as well as by the dynamic terms especially of the model C type. We have calculated the complete field theoretic functions in the two-loop order [34] necessary to calculate the critical (effective) dynamical exponents. Independent of whether the Heisenberg or biconical is the stable static FP the first inspection of the flow of the dynamical parameters shows the following: (i) The imaginary part of the perpendicular relaxation rate renormalizes to zero, (ii) the times scale ratios v (20), $w_\|$ (19) approach zero and the real part of $w_\perp$ still increases. Irrespective of the kind of the stable dynamic FP — whether it is a strong scaling FP with very small but finite values or a weak scaling FP with zero values for v and $w_\|$ — the physical observable features of the magnetic transport coefficient are the effective ones. A range of effective values for the dynamic exponents corresponding to the relaxation of the perpendicular and parallel alternating magnetization and the magnetization is starting around its Van Hove values $z_\perp\sim z_\|\sim z_\lambda\sim 2$ in the background and approach for the biconical FP deep in the asymptotic regime

$$z_\perp\sim 1.6 \qquad z_\|\sim 2 \qquad z_\lambda\sim 1.6\,. \quad (28)$$

The main prediction according to this result would be that the perpendicular and the parallel component of the order parameter would scale differently in this region. *Note added in proof: Further calculations showed that a stable FP can only be reached in the subspace where both* $w_\|\to 0$, $w_\perp\to 0$ *and* $w_\|/w_\perp$ *finite, nonzero. Then* $z_\|=z_\perp\sim 2$ *and* $z_\lambda\sim 1.1$ *for the biconical FP. The results presented here although very near the multicritical point turned out to be only effective exponents. Thus one concludes that the true asymptotics is not reachable in experiments. More details will be published elsewhere.*

The importance of this magnetic system lies in the physical accessibility of the order parameter, contrary to the superfluid ^{4}He or superfluid mixture of ^{4}He and ^{3}He whose dynamics is described by model F [35]. Here all quantities are in principle measurable quantities. Thus the prediction of the different dynamic scaling of the order parameter components can be tested.

IV. CONCLUSIONS AND OUTLOOK

By this review we wanted to summarize recent progress achieved in the theoretical description of the multicritical phenomena. Whereas traditionally RG techniques address critical points in their different realizations, the description of multicritical phenomena is possible both on quantitative and accurate qualitative levels. Moreover, the problem appears to be tractable analytically even if the complicated forms of multicritical dynamics are confronted. As is revealed by the theoretical analysis, a particular feature of static and dynamic behavior inherent to multicritical points is the multitude of fixed points that describe the RG flow. In its turn, this gives rise to a rich effective behavior that may be characterized by different types of multicritical points. A natural continuation of performed studies would be to analyze cumulative effects caused on the multicritical behavior by symmetry breaking factors of different forms (single-ion anisotropies, disorder, frustrations) that might be present in a system.

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ТЕОРЕТИКО-ПОЛЬОВИЙ ПІДХІД ДО АНАЛІЗУ БІКРИТИЧНОЇ ТА ТЕТРАКРИТИЧНОЇ ПОВЕДІНКИ: СТАТИКА І ДИНАМІКА

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За допомогою теоретико-польового ренормгрупового підходу ми досліджуємо статичну й динамічну критичну поведінку тривимірних систем з $O(n_\|)\oplus O(n_\perp)$ симетрією. Тоді як статичні ренормгрупові функції відомі тепер у високих порядках теорії збурень, ми показуємо, що врахування двопетлевих внесків, супроводжене відповідним пересумовуванням асимптотичних рядів, забезпечує акуратний кількісний опис мультикритичної поведінки. Однією із суттєвих рис статичної мультикритичної поведінки, що виявляється вже на двопетлевому рівні для антиферомагнетика в зовнішньому магнітному полі ($n_\| = 1$, $n_\perp = 2$), є стійкість біконічної нерухомої точки та її близькість до меж стійкості інших нерухомих точок. Це приводить до дуже малих значень показників кросоверу. Ми також аналізуємо динамічну критичну поведінку, обираючи різні форми критичної динаміки й обчислюючи у схемі мінімального віднімання асимптотичні та ефективні динамічні критичні показники.